

WHEN AN ABELIAN CATEGORY WITH A TILTING OBJECT IS A MODULE CATEGORY

RICCARDO COLPI, FRANCESCA MANTESE, ALBERTO TONOLO

ABSTRACT. An abelian category with arbitrary coproducts and a small projective generator is equivalent to a module category [23]. A tilting object in an abelian category is a natural generalization of a small projective generator. Moreover, any abelian category with a tilting object admits arbitrary coproducts [12]. It naturally arises the question when an abelian category with a tilting object is equivalent to a module category. By [12] the problem simplifies in understanding when, given an associative ring R and a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in the category of right R -modules, the *heart of the t -structure* $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ associated to $(\mathcal{X}, \mathcal{Y})$ is equivalent to a category of modules. In this paper we give a complete answer to this question, proving necessary and sufficient condition on $(\mathcal{X}, \mathcal{Y})$ for $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ to be equivalent to a module category. We analyze in detail the case when R is right artinian.

INTRODUCTION

In 1964 Barry Mitchell characterized the module categories as those abelian categories with arbitrary coproducts possessing a small and projective generator [23]. At the beginning of the eighties, with the papers of Brenner and Butler, Happel and Ringel, Bongartz and others, the notion of tilting module has been introduced and extensively studied. Tilting modules are small and projective exactly in the subcategory generated by them: they naturally generalize small projective generators.

Tilting theory has been object of further generalizations in the direction of abstract categories, like the case of derived categories [18], [26], Grothendieck categories [9] and abelian categories [20], [11]. In particular in [11] tilting objects for an arbitrary abelian category are defined. Any abelian category with a tilting object admits arbitrary coproducts [12].

Thus, it naturally arises the question when an abelian category with a tilting object is equivalent to a module category. The aim of this paper is to give necessary and sufficient conditions to guarantee such an equivalence.

In [5] Beilinson, Bernstein and Deligne introduced the *heart of a t -structure* in a derived category, proving that it is always an abelian category. In [12] Colpi, Gregorio and Mantese showed that an abelian category with a tilting object V is equivalent to the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ of the t -structure in $\mathcal{D}^b(\text{End } V)$ naturally associated with a suitable faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in the category of right $\text{End } V$ -modules. In the light of this result, our question simplifies in understanding when, given an arbitrary associative ring R , the heart of the t -structure naturally associated with

Date: November 25, 2010.

Research supported by grant CPDA071244/07 of Padova University
2000 Mathematics Subject Classification: 18E10, 18E40, 16D90.

a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-}R$ is equivalent to a module category. This is not always true: for instance the heart associated with the usual torsion pair in the category $\text{Mod-}\mathbb{Z}$ of abelian groups is not equivalent to any module category (see Example 7.1).

Many papers deal with the problem of understanding when the heart of a t -structure is equivalent to a module category in different frameworks (see for example [6], [20], [1], [21], [12], ...). In all these papers, in different ways, a “tilting notion” is always involved.

For example a wide description of the heart associated with a torsion pair is given by Happel, Reiten and Smalø in [20]. In particular they prove, in case of locally finite abelian categories, that if the torsion class is *cogenerating*, i.e., it contains all injective modules, then the heart is equivalent to a module category if and only if the torsion class is generated by a tilting module.

Dually, in our setting, we deal with a faithful torsion pair, that is a torsion pair whose torsion free class contains all projective modules. Following [12] or [21] the heart associated with a faithful torsion pair is equivalent to a module category if and only if it is the heart of a t -structure generated by a tilting complex. Unfortunately this condition is not easily verifiable. In this paper we want to find an explicit characterization in terms of the torsion pair we start from, as Happel, Reiten and Smalø did in [20] in the case the torsion class is cogenerating.

In our main result Theorem 6.1 we give necessary and sufficient conditions on a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-}R$ for $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ to be equivalent to a category of modules. In particular if R is artinian, we prove that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is equivalent to a category of modules if and only if the torsion class \mathcal{X} is generated by a finitely presented R -module V which is tilting in $\text{Mod-}R/\text{Ann}(V)$ (see Corollary 6.2).

In the last section we give concrete examples of how our results apply both in the artinian and not artinian cases. Moreover we give a new proof of the fact that a quasi tilted algebra of finite representation type is tilted, originally proved by Happel and Reiten in [19].

NOTATION

Let \mathcal{C} be an abelian category and V an object of \mathcal{C} . It is possible to associate with V several classes of objects:

- $\text{Gen } V = \{M \in \mathcal{C} : V^{(\alpha)} \rightarrow M \rightarrow 0 \text{ is exact in } \mathcal{C} \text{ for a cardinal } \alpha\}$;
- $\overline{\text{Gen } V}$ is the closure of $\text{Gen } V$ under subobjects;
- $\overline{\text{Gen } \overline{V}}$ is the class of objects in \mathcal{C} which admit a finite filtration with consecutive factors in $\overline{\text{Gen } \overline{V}}$;
- $V^\perp = \text{Ker Ext}_{\mathcal{C}}^1(V, -)$.

Following Dickson [15], a *torsion theory* for \mathcal{C} is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of \mathcal{C} satisfying

- (1) $\mathcal{T} \cap \mathcal{F} = \{0\}$,
- (2) $T \rightarrow A \rightarrow 0$ exact and $T \in \mathcal{T}$ imply $A \in \mathcal{T}$;
- (3) $0 \rightarrow A \rightarrow F$ exact and $F \in \mathcal{F}$ imply $A \in \mathcal{F}$;
- (4) for each $X \in \mathcal{C}$ there is an exact sequence $0 \rightarrow T \rightarrow X \rightarrow F \rightarrow 0$ with $T \in \mathcal{T}$, $F \in \mathcal{F}$.

In such a case \mathcal{T} is a *torsion class* and \mathcal{F} is a *torsion-free class*.

If R is an associative ring with $1 \neq 0$ and M a right R -module, we will denote by R_M the quotient ring $R/\text{Ann}_R(M)$.

1. THE HEART

Given any associative ring R , let $\mathcal{D}^b(R)$ be the bounded derived category of $\text{Mod-}R$. For any complex $M^\bullet \in \mathcal{D}^b(R)$ we denote by $H^i(M^\bullet)$ the i -th cohomology of M^\bullet . If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\text{Mod-}R$, then we denote by $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ the full subcategory of $\mathcal{D}^b(R)$ defined as

$$\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \{M^\bullet \in \mathcal{D}^b(R) \mid H^{-1}(M^\bullet) \in \mathcal{Y}, H^0(M^\bullet) \in \mathcal{X}, H^i(M^\bullet) = 0 \forall i \neq -1, 0\}.$$

$\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is called the *heart* associated with $(\mathcal{X}, \mathcal{Y})$; it is the heart of the t -structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathcal{D}^b(R)$ where

$$\mathcal{D}^{\leq 0} = \{A^\bullet \in \mathcal{D}^b(R) : H^i(A^\bullet) = 0 \text{ for } i > 0, H^0(A^\bullet) \in \mathcal{X}\} \text{ and}$$

$$\mathcal{D}^{\geq 0} = \{A^\bullet \in \mathcal{D}^b(R) : H^i(A^\bullet) = 0 \text{ for } i < -1, H^{-1}(A^\bullet) \in \mathcal{Y}\}$$

In 1982 Beilinson, Bernstein and Deligne [5] proved that the heart is an abelian category.

Remark 1.1. Following [5], a sequence $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is short exact if and only if $A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$ is a triangle in $\mathcal{D}^b(R)$. In particular it follows that given A^\bullet, C^\bullet in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ the Yoneda $\text{Ext}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}^i(C^\bullet, A^\bullet)$ coincides with $\text{Hom}_{\mathcal{D}^b(R)}(C^\bullet, A^\bullet[i])$.

In the sequel, we shortly denote by $M_1 \rightarrow M_0$ the complex

$$\cdots \rightarrow 0 \rightarrow M_1 \rightarrow M_0 \rightarrow 0 \rightarrow \cdots$$

with zero terms everywhere except in degree -1 e 0 . If $(\mathcal{X}, \mathcal{Y})$ is a *faithful* torsion pair in $\text{Mod-}R$, i.e. R belongs to \mathcal{Y} , then each M^\bullet in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is isomorphic to a complex $Y_1 \rightarrow Y_0$ with terms in \mathcal{Y} , which is obtained considering a truncation of a projective resolution of M^\bullet . In particular we have

Lemma 1.2. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in $\text{Mod-}R$ and let $Y_1, Y_0 \in \mathcal{Y}$. An object M^\bullet of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is isomorphic to the complex $Y_1 \xrightarrow{\phi} Y_0$ if and only if the sequence $0 \rightarrow M^\bullet \rightarrow Y_1[1] \xrightarrow{\phi[1]} Y_0[1] \rightarrow 0$ is exact in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

Proof. By Remark 1.1, the sequence

$$0 \rightarrow M^\bullet \rightarrow Y_1[1] \xrightarrow{\phi[1]} Y_0[1] \rightarrow 0$$

is exact in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ if and only if

$$M^\bullet \rightarrow Y_1[1] \xrightarrow{\phi[1]} Y_0[1] \rightarrow M^\bullet[1]$$

is a triangle in $\mathcal{D}^b(R)$, i.e. if and only if

$$M^\bullet \cong \text{cone}(\phi[1])[-1] = Y_1 \xrightarrow{\phi} Y_0.$$

□

In [25] Noohi has given a useful explicit description of morphisms in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Given two objects $M^\bullet := M_1 \rightarrow M_0$ and $N^\bullet := N_1 \rightarrow N_0$ in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, a morphism between M^\bullet and N^\bullet is a isomorphism class of commutative diagrams

$$\begin{array}{ccccc} M_1 & & & & N_1 \\ & \searrow k & & \swarrow \iota & \\ & & E & & \\ & \swarrow \sigma & & \searrow \rho & \\ M_0 & & & & N_0 \end{array}$$

such that the diagonal maps compose to zero and the sequence

$$0 \rightarrow N_1 \xrightarrow{\iota} E \xrightarrow{\sigma} M_0 \rightarrow 0$$

is exact. The kernel and cokernel of this morphism are given by the complexes

$$M_1 \xrightarrow{k} A \quad \text{and} \quad E/A \xrightarrow{\rho} N_0$$

where A is the unique submodule of E sitting between $\text{Im } k$ and $\text{Ker } \rho$ such that $A/\text{Im } k \in \mathcal{X}$ and $\text{Ker } \rho/A \in \mathcal{Y}$.

2. THE PROBLEM

Happel, Reiten and Smalø in [20] have introduced the notion of tilting object in locally finite abelian categories, generalizing that of *classical 1-tilting* module, i.e., a finitely generated tilting module of projective dimension ≤ 1 . Colpi and Fuller in [11] have further generalized this notion for an arbitrary abelian category:

Definition 2.1. [11, Definition 2.3] *An object V in an abelian category \mathcal{C} is called tilting if:*

- (1) \mathcal{C} contains arbitrary coproducts of copies of V ;
- (2) V is selfsmall (i.e., $\text{Hom}_{\mathcal{C}}(V, V^{(\alpha)}) \cong (\text{End } V)^{(\alpha)}$ for any cardinal α);
- (3) $\text{Gen } V = V^\perp$;
- (4) $\overline{\text{Gen}} V = \mathcal{C}$.

By [11, Proposition 2.1] a tilting object has projective dimension ≤ 1 .

Possessing a tilting object is a very tightening condition for an abelian category. In particular it is an AB4 category, i.e. it has arbitrary and exact coproducts (see [12, Lemma 3.2]). On the other hand, an object V in an AB4 category is tilting if and only if it satisfies conditions (2), (3) in Definition 2.1 (see [11, Remark 2.2]).

A tilting object V in an abelian category \mathcal{C} generates a torsion pair $(\mathcal{T} = \text{Gen } V, \mathcal{F})$ which is counter equivalent (see [8] and [11]) to a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-End } V$, called the *tilted torsion pair* of $(\mathcal{T}, \mathcal{F})$. Precisely the torsion class \mathcal{X} coincides with the image in $\text{Mod-End } V$ of the functor $\text{Ext}_{\mathcal{C}}^1(V, -)$ and the torsion-free class \mathcal{Y} coincides with the image in $\text{Mod-End } V$ of the functor $\text{Hom}_{\mathcal{C}}(V, -)$.

Example 2.2. *If $(\mathcal{X}, \mathcal{Y})$ is a faithful torsion pair in $\text{Mod-}R$, the complex $R[1]$ is a tilting object in the abelian category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. It is $\text{Gen}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})} R[1] = \mathcal{Y}[1]$ and $(\mathcal{Y}[1], \mathcal{X})$ is a torsion pair in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ naturally counter equivalent to the torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-End } R[1] = \text{Mod-}R$. The counter equivalence between $(\mathcal{Y}[1], \mathcal{X})$ and $(\mathcal{X}, \mathcal{Y})$ is given by the functors*

$$H := \text{Hom}_{\mathcal{H}}(R[1], -) : \mathcal{H}(\mathcal{X}, \mathcal{Y}) \rightarrow \text{Mod-}R, \text{ and}$$

$$H' := \text{Ext}_{\mathcal{H}}^1(R[1], -) : \mathcal{H}(\mathcal{X}, \mathcal{Y}) \rightarrow \text{Mod-}R$$

and by their adjoint functors T and T' . For any complex M^\bullet in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ and any module N in $\text{Mod-}R$, denoted by $t_{\mathcal{X}}$ the radical of the torsion pair $(\mathcal{X}, \mathcal{Y})$, it is

$$H(M^\bullet) = H^{-1}(M^\bullet), H'(M^\bullet) = H^0(M^\bullet), T(N) = (N/t_{\mathcal{X}}(N))[1], T'(N) = t_{\mathcal{X}}(N)[0].$$

The last is much more than an example. Indeed, Colpi, Gregorio and Mantese proved

Theorem 2.3. [12, Corollary 2.4] *Let \mathcal{C} be an abelian category with a tilting object V , and $(\mathcal{T}, \mathcal{F})$ the torsion pair generated by V . Then the category \mathcal{C} is equivalent to the heart associated with the tilted torsion pair $(\mathcal{X}, \mathcal{Y})$ of $(\mathcal{T}, \mathcal{F})$ in $\text{Mod-End } V$. Moreover $(\text{End } V)[1]$ is the tilting object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ corresponding to V by the equivalence.*

In 1964 Barry Mitchell characterized the module categories as those abelian categories with arbitrary coproducts possessing a small and projective generator [23]. Since the tilting objects are a natural generalization of small projective generators, it is natural to place the following

Problem: when an abelian category \mathcal{A} with a tilting object V is equivalent to a module category?

By the above quoted result of Colpi, Gregorio and Mantese this problem is equivalent to understand when the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ associated with a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ is equivalent to a category of modules.

3. QUASI-TILTING AND TILTING MODULES

Let R be an associative ring. Applying Definition 2.1 to $\mathcal{A} = \text{Mod-}R$ we have that a right R -module V is *tilting* if $\text{Gen } V = V^\perp$ or equivalently (see [14, Proposition 1.3]) if

- (T1) there exists a short exact sequence $0 \rightarrow R_1 \rightarrow R_0 \rightarrow V \rightarrow 0$ with R_0, R_1 direct summands of a finite direct sum of copies of R ;
- (T2) $\text{Ext}_R^1(V, V) = 0$;
- (T3) there exists a short exact sequence $0 \rightarrow R \rightarrow V_0 \rightarrow V_1 \rightarrow 0$ with V_0, V_1 direct summands of a finite direct sum of copies of V .

Let us emphasize that this notion of “tilting module” corresponds in the recent literature to that of “classical 1-tilting module” [17, Definition 5.1.1].

In [10] the following generalization of the tilting notion has been studied:

Definition 3.1 (Definition 2.2, [10]). *A right R -module V is called quasi-tilting if it is finitely generated and*

$$\text{Gen } V_R = \overline{\text{Gen } V_R} \cap V_R^\perp$$

Quasi-tilting modules represent the equivalences between a torsion class and a torsion-free class in categories of modules (see [10, Theorem 2.6]. They are an effective generalization of the tilting notion: in [10, Proposition 2.3] it is proved that a quasi-tilting module V_R is a tilting module if and only if it is faithful and $\text{Gen } V$ is closed under products.

Given a right module V we will denote by R_V the quotient ring $R/\text{Ann } V$.

Proposition 3.2. *Let V_R be a right R -module.*

- (1) *If V is a quasi-tilting R -module and $\text{Gen } V_R$ is closed under products, then V is a tilting R_V -module.*
- (2) *If V is a tilting R_V -module and $\text{Gen } V_R$ is closed under extensions, then V is a quasi-tilting R -module.*

Proof. First we observe that $\text{Pres } V_{R_V} = \text{Pres } V_R$, $\text{Gen } V_{R_V} = \text{Gen } V_R$ and V_{R_V} is finitely generated if and only if V_R is finitely generated.

1. Since $V_R^\perp \cap \text{Mod-}R_V \subseteq V_{R_V}^\perp$, by [10, Proposition 2.1.(iii)] V is also a quasi-tilting right R_V -module. By [10, Proposition 2.3.(iv)] if $\text{Gen } V_R$ is closed under products, then V is a tilting R_V -module.

2. By [10, Proposition 2.1.(iii)], we have to prove that $\text{Gen } V_R \subseteq V_R^\perp$. Consider a short exact sequence in $\text{Mod-}R$

$$0 \rightarrow X \rightarrow Z \rightarrow V \rightarrow 0;$$

since $\text{Gen } V_R$ is closed under extensions, the right R -module Z belongs to $\text{Gen } V_R$. Then $0 \rightarrow X \rightarrow Z \rightarrow V \rightarrow 0$ is also a short exact sequence in $\text{Mod-}R_V$; since V is a tilting R_V -module, the sequence splits. \square

4. NECESSARY CONDITIONS

In this section we will give some necessary conditions for the heart associated with a faithful torsion pair to be equivalent to a whole category of modules.

Lemma 4.1. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in $\text{Mod-}R$ and assume that there is an equivalence*

$$\mathcal{H}(\mathcal{X}, \mathcal{Y}) \xleftarrow{*} \text{Mod-}S$$

between the heart associated with $(\mathcal{X}, \mathcal{Y})$ and the category of right S -modules for a suitable associative ring S . Then there exists a finitely presented right R -module V generating \mathcal{X} such that

- (1) *V is a R_V -tilting module;*
- (2) *the ring S corresponds by the equivalence to a complex $R_1 \xrightarrow{f} R_0$, where R_1, R_0 are finitely generated projective R -modules and $\text{Coker } f = V$.*

Proof. Let us denote by U_S the right S -module $R[1]^*$; since $R[1]$ is a tilting object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, U_S is a tilting S -module. Clearly $R \cong \text{End } R[1] \cong \text{End } U_S$. Therefore by [24] also ${}_R U$ is a tilting module, in particular it is finitely presented. Let $(\mathcal{T} = \text{Gen } U_S, \mathcal{F})$ be the torsion pair generated by U_S ; composing the natural counter equivalence between $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-}R$ and $(\mathcal{Y}[1], \mathcal{X})$ in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ (see Example 2.2) with the equivalence $\mathcal{H}(\mathcal{X}, \mathcal{Y}) \xleftarrow{*} \text{Mod-}S$, we get a counter equivalence $\mathcal{T} \leftrightarrow \mathcal{Y}$ and $\mathcal{F} \leftrightarrow \mathcal{X}$ induced by ${}_R U_S$ via the $\text{Hom}_S(U_S, -)$ and $- \otimes_R U$ functors and their first derived functors $\text{Ext}_S^1(U, -)$ and $\text{Tor}_1^R(-, U)$. It is $\text{Hom}_S(U, M) = H^{-1}(M^*)$ and $N \otimes_R U = ((N/t_{\mathcal{X}}(N))[1])^*$ (see Example 2.2).

(1) By [10, Theorems 2.6, 3.4], $V_R := \text{Ext}_S^1({}_R U_S, S)$ is a quasi tilting module which generates \mathcal{X} . Since $\mathcal{X} = \text{Ker } - \otimes_R U$ and ${}_R U$ is finitely presented, it is closed under products; by Proposition ?? V is a tilting R_V -module by Proposition 3.2.

(2) By property (T3) of tilting modules, there exists a short exact sequence

$$0 \rightarrow S \rightarrow U_1 \rightarrow U_0 \rightarrow 0$$

with U_1, U_0 direct summands of a finite direct sum of copies of U_S . Applying $\text{Hom}_S(U, -)$ we get

$$0 \rightarrow \text{Hom}_S(U, S) \rightarrow R_1 \rightarrow R_0 \rightarrow V \rightarrow 0$$

with the R_i 's finitely generated projective R -modules. In particular V_R is finitely presented.

Since $\text{Hom}_S(U, S)$ belongs to \mathcal{Y} and V belongs to \mathcal{X} , the complex $R_1 \rightarrow R_0$ belongs to the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Let us see that S^* is isomorphic to the complex $R_1 \rightarrow R_0$. Since $U_i \cong R_i \otimes_R U = R_i[1]^*$, we have $U_i^* \cong R_i[1]$. Applying the equivalence $*$ to $0 \rightarrow S \rightarrow U_1 \rightarrow U_0 \rightarrow 0$ we get

$$0 \rightarrow S^* \rightarrow R_1[1] \rightarrow R_0[1] \rightarrow 0.$$

By Lemma 1.2, S^* is isomorphic to the complex $R_1 \rightarrow R_0$. \square

In order to understand when the heart of a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-}R$ is equivalent to a module category, by Lemma 4.1 (1), we can assume that $\mathcal{X} = \text{Gen } V_R$ where V is a finitely presented R -module and a tilting R_V -module.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ TO BE A MODULE CATEGORY

Let us assume $(\mathcal{X}, \mathcal{Y})$ to be a faithful torsion pair in $\text{Mod-}R$ with $\mathcal{X} = \text{Gen } V$ where V_R is finitely presented and a tilting R_V -module. Since the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ admits arbitrary coproducts [12, Lemma 3.1], by [23, Theorem 3.1] it is equivalent to a module category if and only if it has a *small projective generator*. By Lemma 4.1 (2), we can look for a small projective generator among the finitely generated projective presentations of V in $\text{Mod-}R$.

In the sequel, therefore, we will investigate necessary and sufficient conditions on a finitely generated projective presentation $R_1 \xrightarrow{f} R_0 \rightarrow V_R \rightarrow 0$ of the tilting R_V -module V which generates \mathcal{X} for the complex $R_1 \xrightarrow{f} R_0$ to be a small projective generator of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.

5.1. The smallness of $R_1 \xrightarrow{f} R_0$. It is easy to verify that :

Lemma 5.1. *Let $R_1 \rightarrow R_0$ be in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, with R_0, R_1 finitely generated projectives in $\text{Mod-}R$. Then $R_1 \rightarrow R_0$ is small.*

Proof. In [11, Lemma 4.1] it is shown that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is closed under coproducts in $\mathcal{D}^b(R)$ and that coproducts are defined componentwise. Let us prove that any morphism

$$\phi : (R_1 \rightarrow R_0) \rightarrow \coprod_{\lambda \in \Lambda} M_\lambda^\bullet$$

in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ factorizes through a finite coproduct $\coprod_{\lambda \in \Lambda_0} M_\lambda^\bullet$ for a finite subset Λ_0 of Λ . Since R_0, R_1 are projective R -modules, ϕ is a morphism in the homotopic category. We conclude since R_0, R_1 are finitely generated modules. \square

5.2. When $R_1 \xrightarrow{f} R_0$ is projective. We give now necessary and sufficient conditions for the complex $P^\bullet := R_1 \xrightarrow{f} R_0$ to be a projective object in the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. The result is valid for any complex $M_1 \rightarrow M_0$ in the heart with projective terms.

Proposition 5.2. *The complex $P^\bullet := R_1 \xrightarrow{f} R_0$ is projective in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ if and only if $\text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1]) = 0$, i.e. for any map $\varphi : R_1 \rightarrow V$ there exists a map $\psi : R_0 \rightarrow V$ such that $\varphi = \psi f$.*

Proof. If P^\bullet is projective in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, then we have by Remark 1.1

$$\text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1]) = \text{Ext}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}^1(P^\bullet, V) = 0.$$

Conversely, assume $\text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1]) = 0$. Since $(\mathcal{Y}[1], \mathcal{X})$ is a torsion pair in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, in order to prove that P^\bullet is projective it is enough to check that $\text{Ext}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}^1(P^\bullet, \mathcal{Y}[1]) = 0 = \text{Ext}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}^1(P^\bullet, \mathcal{X})$. By Lemma 1.2

$$0 \rightarrow P^\bullet \rightarrow R_1[1] \rightarrow R_0[1] \rightarrow 0$$

is an exact sequence in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.

The objects $R_1[1]$ and $R_0[1]$ belong to $\text{Add}(R[1])$ and hence they have projective dimension ≤ 1 ; hence first we have $\text{proj dim}(P^\bullet) \leq 1$.

Since $R[1]$ is tilting and $\text{Gen}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})} R[1] = \mathcal{Y}[1]$, by Definition 2.1 it is $\text{Add}(R[1]) \subseteq \text{Ker Ext}^1(-, \mathcal{Y}[1])$; therefore we get $\text{Ext}^1(P^\bullet, \mathcal{Y}[1]) = 0$.

Let now X be a right R -module in $\mathcal{X} = \text{Gen } V$; consider an epimorphism $V^{(\alpha)} \rightarrow X \rightarrow 0$. It is also an epimorphism between stalk complexes in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Since

$$\begin{aligned} \text{Ext}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}^1(P^\bullet, V^{(\alpha)}) &= \text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1]^{(\alpha)}) \subseteq \\ &\subseteq \text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1]^\alpha) = \text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1]^\alpha) = 0, \end{aligned}$$

and $\text{proj dim}(P^\bullet) \leq 1$, we conclude that $\text{Ext}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}^1(P^\bullet, X) = 0$. \square

Remark 5.3. By Proposition 5.2 the complex $R_1 \xrightarrow{f} R_0$ is projective in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ if and only if

$$\phi(\text{Ker } f) = 0 \quad \text{for each } \phi \in \text{Hom}(R_1, V).$$

Since $R_1 \leq^\oplus R^m$ for a suitable $m \in \mathbb{N}$, and

$$\bigcap_{\phi \in \text{Hom}(R^m, V)} \text{Ker } \phi = (\text{Ann}_R V)^m$$

we have that $R_1 \xrightarrow{f} R_0$ is projective if and only if $\text{Ker } f$ as submodule of R^m is contained in $(\text{Ann}_R V)^m$. This condition suggests to choose a presentation $R_1 \xrightarrow{f} R_0$ of V with $\text{Ker } f$ as small as possible.

The next result goes in the same direction of the above remark. But first let us recall the following useful classical homological result.

Lemma 5.4. [22, Lemma B.1] *Consider the following diagram in $\text{Mod-}R$ with exact rows*

$$\begin{array}{ccccccc} C & \xrightarrow{f} & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ \downarrow h & \swarrow p & \downarrow & \searrow q & \downarrow \ell & & \\ 0 \longrightarrow & L & \longrightarrow & M & \xrightarrow{g} & N & \end{array}$$

There exists $q : C'' \rightarrow M$ such that $g \circ q = \ell$ if and only if there exists $p : C' \rightarrow L$ such that $p \circ f = h$.

Proposition 5.5. *If the module R_1 in the complex $P^\bullet := R_1 \xrightarrow{f} R_0$ is a projective cover of $\text{Im } f$, then P^\bullet is a projective object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.*

Proof. Let ϕ be a morphism in $\text{Hom}_{\mathcal{D}^b(R)}(P^\bullet, V[1])$ and denote by Ω the kernel of $f : R_1 \rightarrow R_0$. Consider the following diagram in $\text{Mod-}R$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega & \xrightarrow{j} & R_1 & \xrightarrow{f} & R_0 & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow \hat{\phi} & & \downarrow \phi & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \phi(\Omega) & \longrightarrow & V & \xrightarrow{g} & Q & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

where Q is the pushout of the maps f and ϕ . Since $\text{Im } g$ belongs to $\text{Gen } V \subseteq V^\perp$ (see Proposition 3.2.(2)), the sequence

$$0 \rightarrow \text{Im } g \rightarrow Q \rightarrow V \rightarrow 0$$

splits. Therefore by Lemma 5.4 and the projectivity of R_0 we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega & \xrightarrow{j} & R_1 & \xrightarrow{f} & R_0 & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow \hat{\phi} & & \downarrow \phi & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \phi(\Omega) & \longrightarrow & V & \xrightarrow{g} & Q & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

$\nearrow \hat{g}$ $\nwarrow k$ $\nwarrow \hat{k}$ $\nwarrow \hat{g}$
 $\text{Im } g$

such that $\hat{g} \circ \phi = \hat{g} \circ k \circ f$. Let us denote by \hat{k} the restriction of k to $\text{Im } f$; applying again Lemma 5.4 we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega & \xrightarrow{j} & R_1 & \xrightarrow{\hat{f}} & \text{Im } f & \longrightarrow & 0 \\ & & \downarrow \hat{\phi} & & \downarrow \phi & & \downarrow & & \\ 0 & \longrightarrow & \phi(\Omega) & \longrightarrow & V & \xrightarrow{\hat{g}} & \text{Im } g & \longrightarrow & 0 \end{array}$$

$\nearrow \theta$ $\nwarrow \hat{k}$
 $\text{Im } g$

with $\theta \circ j = \hat{\phi}$. Therefore, since $\text{Im } \theta = \text{Im } \hat{\phi}$, we have $R_1 = \text{Im } j + \text{Ker } \theta$; since $\text{Im } j$ is superfluous in R_1 , we get $\text{Ker } \theta = R_1$, i.e. $\hat{\phi} = \theta \circ j = 0$. Thus the map \hat{g} is a monomorphism and from $\hat{g} \circ \phi = \hat{g} \circ k \circ f$ we get $\phi = k \circ f$. We conclude by Proposition 5.2. \square

5.3. When $R_1 \xrightarrow{f} R_0$ is a generator. Assume $R_1 \xrightarrow{f} R_0 \rightarrow V \rightarrow 0$ is a projective presentation of V such that the complex $P^\bullet = R_1 \rightarrow R_0$ is a projective object in the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Continue to denote by Ω the kernel of f .

In this section we give necessary and sufficient conditions for P^\bullet to be a generator in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Let us start with some preliminary lemmas.

Lemma 5.6. *Let M be a right R -module and assume $\text{Gen } M = \text{Pres } M$ and closed under extensions. Consider a module L in $\overline{\text{Gen}} M$; then*

(1) *there exists a short exact sequence*

$$0 \rightarrow L \rightarrow X \rightarrow M^{(\alpha)} \rightarrow 0$$

with $X \in \text{Gen } M$;

(2) *any extension of $M^{(\alpha)}$ by L belongs to $\overline{\text{Gen}} M$.*

Proof. There exists a short exact sequence

$$0 \rightarrow L \xrightarrow{\iota} X_1 \rightarrow X_2 \rightarrow 0$$

with X_1, X_2 belonging to $\text{Gen } M$.

(1) The module X_2 is an homomorphic image of $M^{(\alpha)}$ for a suitable cardinal α . Then we have the following diagram with exact rows and columns

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & L & \longrightarrow & X_1 & \xrightarrow{p} & X_2 \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \pi \\ 0 & \longrightarrow & L & \longrightarrow & X & \longrightarrow & M^{(\alpha)} \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & M^{(\beta)} & \xlongequal{\quad} & M^{(\beta)} \end{array}$$

where X is the pullback of the maps p and π . Clearly X results to be an extension of modules in $\text{Gen } M$, and therefore it belongs to $\text{Gen } M$.

(2) Let Y be any extension of $M^{(\alpha)}$ by L . We have the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\psi} & Y & \longrightarrow & M^{(\alpha)} \longrightarrow 0 \\ & & \downarrow \iota & & \downarrow & & \parallel \\ 0 & \longrightarrow & X_1 & \longrightarrow & Q & \longrightarrow & M^{(\alpha)} \longrightarrow 0 \end{array}$$

where Q is the pushout of ι and ψ . Since $\text{Gen } M$ is closed under extensions, we have that Q belongs to $\text{Gen } M$ and hence Y is subgenerated by M . \square

Let us denote by $\overline{\overline{\text{Gen}}} V$ the class of right R -modules M which admit a finite filtration

$$M = M_0 \geq M_1 \geq \dots \geq M_k = 0, \quad k \in \mathbb{N}$$

with $M_i/M_{i+1} \in \overline{\text{Gen}} V$ for $i = 0, \dots, k-1$.

Lemma 5.7. *The class $\overline{\overline{\text{Gen}}} V$ is closed under submodules, quotients and extensions.*

Proof. Clearly $\overline{\overline{\text{Gen}}} V$ is closed under extensions. Let us see that $\overline{\overline{\text{Gen}}} V$ is closed under submodules and quotients. Let M be in $\overline{\overline{\text{Gen}}} V$; consider the filtration

$$M = M_0 \geq M_1 \geq \dots \geq M_k = 0, \quad k \in \mathbb{N}$$

with $M_i/M_{i+1} \in \overline{\text{Gen}} V$ for $i = 0, \dots, k-1$. We prove the closure by induction on the length k of the filtration. If $k = 1$, then M belongs to $\overline{\text{Gen}} V$ and the latter

is closed under submodules and quotients. Let $k > 1$ and $L \leq M$. Consider the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\varepsilon} & M & \longrightarrow & M/M_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow \iota & & \parallel \\ 0 & \longrightarrow & L \cap M_1 & \longrightarrow & L & \longrightarrow & (L + M_1)/M_1 \longrightarrow 0 \end{array}$$

By the inductive hypothesis, since M_1 has a filtration of length $k - 1$ and M/M_1 belongs to $\overline{\text{Gen}}V$, the modules $L \cap M_1$ and $(L + M_1)/M_1$ belong to $\overline{\text{Gen}}V$. Then we conclude that L belongs to $\overline{\text{Gen}}V$. Consider now the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_{k-1} & \longrightarrow & M & \xrightarrow{q} & M/M_{k-1} \longrightarrow 0 \\ & & \parallel & & \downarrow p & & \downarrow \\ & & M_{k-1} & \longrightarrow & M/L & \longrightarrow & N \longrightarrow 0 \end{array}$$

where N is the pushout of p and q . Since $M/M_{k-1} \in \overline{\text{Gen}}V$ has a filtration of length $k - 1$, by induction N belongs to $\overline{\text{Gen}}V$; then M/L is an extension of N by a homomorphic image of $M_{k-1} \in \overline{\text{Gen}}V$ and hence it belongs to $\overline{\text{Gen}}V$. \square

We continue to denote by T , as in the Example 2.2, the functor $\text{Mod-}R \rightarrow \mathcal{H}(\mathcal{X}, \mathcal{Y})$ which associates with a module N_R the complex $N/t_{\mathcal{X}}(N)[1]$ where $t_{\mathcal{X}}$ is the radical of the torsion pair $(\mathcal{X}, \mathcal{Y})$.

Lemma 5.8. *Let $L \in \mathcal{Y} \cap \overline{\text{Gen}}V$. Then $T(L) = L[1]$ belongs to $\text{Gen } P^\bullet$.*

Proof. The module L has a finite filtration

$$L = L_0 \geq L_1 \geq \dots \geq L_k = 0, \quad k \in \mathbb{N}$$

with $L_i/L_{i+1} \in \overline{\text{Gen}}V$ for $i = 0, \dots, k - 1$. We prove the claim by induction on the length k of the filtration of L .

$k = 1$: Since V is a tilting R_V -module, then $\text{Gen } V = \text{Pres } V$ and the latter is closed under extensions. Therefore we can apply Lemma 5.6.(1) to get a short exact sequence

$$0 \rightarrow N \rightarrow X \rightarrow V^{(\alpha)} \rightarrow 0$$

with $X \in \text{Gen } V$. Applying the functor T we get $T'V^{(\alpha)} \rightarrow TN \rightarrow TX = 0$. It is easy to see that $T'(V^{(\alpha)})$ is generated by P^\bullet : indeed by [25, Corollary 3.3] the following is an epimorphism on the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$:

$$P^{\bullet(\beta)} = \begin{array}{ccc} R_1^{(\beta)} & & 0 \\ \downarrow & \longrightarrow & \downarrow \\ R_0^{(\beta)} & & V^{(\beta)} \end{array}$$

Therefore also TN belongs to $\text{Gen } P^\bullet$.

$k > 1$: We have the following diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L_1 & \longrightarrow & Q & \longrightarrow & t(L/L_1) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow i \\
 0 & \longrightarrow & L_1 & \longrightarrow & L & \xrightarrow{p} & L/L_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & K & \xlongequal{\quad} & K
 \end{array}$$

where $t(L/L_1)$ is the torsion part of L/L_1 and Q is the pullback of i and p . Since L_1 has a filtration of length $k - 1$, by inductive hypothesis $TL_1 \in \text{Gen } P^\bullet$. Therefore applying T to the first row of the above diagram we get that $TQ \in \text{Gen } P^\bullet$. Now K is the torsion free part of L/L_1 and therefore it belongs to \mathcal{Y} ; since $L/L_1 \in \overline{\text{Gen}} V$, the module K belongs also to $\overline{\text{Gen}} V$. By Lemma 5.8 we have $TK \in \text{Gen } P^\bullet$. Since Q, L, K belong to \mathcal{Y} we have the following short exact sequence in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$

$$0 \rightarrow TQ \rightarrow TL \rightarrow TK \rightarrow 0$$

By a standard argument, since P^\bullet is projective, $\text{Gen } P^\bullet$ is closed under extensions and so TL belongs to $\text{Gen } P^\bullet$. \square

Proposition 5.9. *The projective object $P^\bullet := R_1 \xrightarrow{f} R_0$ is a generator of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ if and only if there exists a cardinal α and a morphism $g \in \text{Hom}_R(R_1^{(\alpha)}, R)$ such that the cokernel of the restriction $g|_{\Omega^{(\alpha)}}$ belongs to $\overline{\text{Gen}} V$.*

Proof. Let assume that P^\bullet is a projective generator. Then there is an exact sequence in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ of the form

$$(*) \quad 0 \rightarrow K^\bullet \rightarrow P^{\bullet(\alpha)} \xrightarrow{\phi} R[1] \rightarrow 0$$

Here ϕ is a complex map from $P^{\bullet(\alpha)} = R_1^{(\alpha)} \xrightarrow{f^{(\alpha)}} R_0^{(\alpha)}$ to $R[1]$. Clearly only the component on degree -1 of ϕ is different from 0: it is the wanted morphism g in $\text{Hom}_R(R_1^{(\alpha)}, R)$. Indeed, the exact sequence $(*)$ induces the long exact sequences of homologies

$$H^{-1}P^{\bullet(\alpha)} = \Omega^{(\alpha)} \xrightarrow{H^{-1}\phi} H^{-1}R[1] = R \rightarrow H^0K^\bullet$$

Since H^0K^\bullet belongs to \mathcal{X} , and $H^{-1}\phi$ is the restriction of g at $\Omega^{(\alpha)}$, we get that the cokernel of the restriction $g|_{\Omega^{(\alpha)}}$ belongs to $\overline{\text{Gen}} V$.

Conversely, assume the existence of the cardinal α and the map g in $\text{Hom}_R(R_1^{(\alpha)}, R)$ such that the cokernel of the restriction $g|_{\Omega^{(\alpha)}}$ belongs to $\overline{\text{Gen}} V$. First notice that in order to prove that P^\bullet is a generator it is enough to prove that P^\bullet generates the tilting object $R[1]$. Indeed, by construction, there is an epimorphism in the heart $P^\bullet \rightarrow V$ and hence P^\bullet generates the torsion free class $\mathcal{X} = \text{Gen } V$; if P^\bullet generates also $R[1]$, then it generates the torsion class $\mathcal{Y}[1]$ and so, being projective, also the extensions of \mathcal{X} by $\mathcal{Y}[1]$, i.e. the whole category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.

Let us consider the following map in the heart:

$$P^\bullet(\alpha) \xrightarrow{\mu} R[1] \quad := \quad \begin{array}{ccc} R_1^{(\alpha)} & \xrightarrow{g} & R \\ \downarrow f^{(\alpha)} & & \downarrow \\ R_0^{(\alpha)} & \longrightarrow & 0 \end{array}$$

Let us denote by h the map

$$R_1^{(\alpha)} \xrightarrow{(g, -f^{(\alpha)})} R \oplus R_0^{(\alpha)}.$$

Then we have the following diagram with exact rows and columns in $\text{Mod-}R$:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & g(\text{Ker } f^{(\alpha)}) = g(\Omega^{(\alpha)}) & \longrightarrow & R & \longrightarrow & R/g(\Omega^{(\alpha)}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Im } h & \longrightarrow & R \oplus R_0^{(\alpha)} & \longrightarrow & R \oplus R_0^{(\alpha)} / \text{Im } h \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Im } f^{(\alpha)} & \longrightarrow & R_0^{(\alpha)} & \longrightarrow & V^{(\alpha)} \longrightarrow 0 \end{array}$$

By hypothesis $R/g(\Omega^{(\alpha)})$ belongs to $\overline{\text{Gen}}V$; therefore also $R \oplus R_0^{(\alpha)} / \text{Im } h$ belongs to $\overline{\text{Gen}}V$ by Lemma 5.7. Let $A/\text{Im } h$ with $\text{Im } h \leq A \leq R \oplus R_0^{(\alpha)}$ the torsion part of $R \oplus R_0^{(\alpha)} / \text{Im } h$; then $R \oplus R_0^{(\alpha)} / A$ belongs to \mathcal{Y} . By the following diagram and Lemma 5.7 it belongs also to $\overline{\text{Gen}}V$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R/g(\Omega^{(\alpha)}) & \longrightarrow & R \oplus R_0^{(\alpha)} / \text{Im } h & \xrightarrow{p} & V^{(\alpha)} \longrightarrow 0 \\ & & \parallel & & \downarrow q & & \downarrow \\ & & R/g(\Omega^{(\alpha)}) & \longrightarrow & R \oplus R_0^{(\alpha)} / A & \longrightarrow & X \longrightarrow 0 \end{array}$$

where X is the pushout of p and q . By Lemma 5.8 $T(R \oplus R_0^{(\alpha)} / A)$ belongs to $\text{Gen } P^\bullet$. Following [25, section 3.4], $(R \oplus R_0^{(\alpha)} / A)[1]$ is the cokernel in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ of $P^\bullet(\alpha) \xrightarrow{\mu} R[1]$. Thus we have the following diagram with exact row in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$:

$$\begin{array}{ccccc} P^\bullet(\alpha) & \xrightarrow{\mu} & R[1] & \longrightarrow & (R \oplus R_0^{(\alpha)} / A)[1] \longrightarrow 0 \\ & & \nwarrow \lambda & & \uparrow \\ & & & & P^\bullet(\beta) \end{array}$$

where λ is obtained by the projectivity of $P^\bullet(\beta)$. The map $\lambda \oplus \mu : P^\bullet(\beta) \oplus P^\bullet(\alpha) \rightarrow R[1]$ is an epimorphism in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, and hence $R[1]$ is generated by P^\bullet . \square

Remark 5.10. The previous result suggests to consider a presentation $R_1 \xrightarrow{f} R_0$ of V with $\Omega = \text{Ker } f$ as big as possible to get a map $g : R_1^{(\alpha)} \rightarrow R$ with $\text{Coker } g|_{\Omega(\alpha)}$ belonging to $\overline{\text{Gen}}V$, going in the opposite direction with what we have observed in Remark 5.3.

Let $\mathcal{V} := \{V/K : K \leq V\}$. For any right R -module M , consider

$$\text{Rej}_{\mathcal{V}} M = \bigcap_{f:M \rightarrow W, W \in \mathcal{V}} \text{Ker } f.$$

Clearly we have the following *reject chain* of M :

$$M \geq \text{Rej}_{\mathcal{V}} M \geq \text{Rej}_{\mathcal{V}}^2 M := \text{Rej}_{\mathcal{V}}(\text{Rej}_{\mathcal{V}} M) \geq \text{Rej}_{\mathcal{V}}^3 M \geq \text{Rej}_{\mathcal{V}}^4 M \geq \dots$$

Each factor $\text{Rej}_{\mathcal{V}}^i M / \text{Rej}_{\mathcal{V}}^{i+1} M$ belongs to $\overline{\text{Gen}}V$, for

$$\text{Rej}_{\mathcal{V}}^i M / \text{Rej}_{\mathcal{V}}^{i+1} M \hookrightarrow \prod_{W \in \mathcal{V}} W^{\text{Hom}(\text{Rej}_{\mathcal{V}}^i M, W)},$$

$$m + \text{Rej}_{\mathcal{V}}^{i+1} M \mapsto (\varphi_W(m))_{\varphi_W \in \text{Hom}(\text{Rej}_{\mathcal{V}}^i M, W), W \in \mathcal{V}}$$

Therefore for any $i \in \mathbb{N}$, the module $M / \text{Rej}_{\mathcal{V}}^i M$ belongs to $\overline{\text{Gen}}V$.

Proposition 5.11. *Assume the chain*

$$R \geq \text{Rej}_{\mathcal{V}} R \geq \text{Rej}_{\mathcal{V}}^2 R \geq \text{Rej}_{\mathcal{V}}^3 R \geq \dots$$

is stationary, and $\text{Rej}_{\mathcal{V}}^n R = \bigcap_{i \in \mathbb{N}} \text{Rej}_{\mathcal{V}}^{n+i} R$ admits a projective cover $\xi : R_2 \rightarrow \text{Rej}_{\mathcal{V}}^n R$. Then $R_1 \oplus R_2 \xrightarrow{(f,0)} R_0$ is a projective generator of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.

Proof. First let us prove that $\text{Hom}_R(R_2, V) = 0$. Given $\zeta \in \text{Hom}_R(R_2, V)$, consider the following diagram with exact rows:

$$\begin{array}{ccccc} R_2 & \xrightarrow{\xi} & \text{Rej}_{\mathcal{V}}^n R & \longrightarrow & 0 \\ \downarrow \zeta & & \downarrow \pi_2 & & \\ V & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

where $Q := V \oplus \text{Rej}_{\mathcal{V}}^n R / \langle (\zeta(m), \xi(m)) : m \in R_2 \rangle$ is the pushout of ξ and ζ . Since $\text{Rej}_{\mathcal{V}}^n R = \text{Rej}_{\mathcal{V}}^{n+1} R$, $\pi_2 = 0$ and hence for each $\ell \in R_2$ it is $\pi_2(\xi(\ell)) = 0$, i.e. $(0, \xi(\ell)) \in \langle (\zeta(m), \xi(m)) : m \in R_2 \rangle$. Therefore, for each $\ell \in R_2$ there exists $m_\ell \in \text{Ker } \zeta$ such that $\xi(m_\ell) = \xi(\ell)$. Thus we have $R_2 = \text{Ker } \zeta + \text{Ker } \xi$; since $\text{Ker } \xi$ is superfluous, we have $R_2 = \text{Ker } \zeta$ and hence $\zeta = 0$. Now since $\text{Hom}_R(R_1 \oplus R_2, V) = \text{Hom}_R(R_1, V)$ and $R_1 \xrightarrow{f} R_0$ is a projective object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, by Proposition 5.2 also $R_1 \oplus R_2 \xrightarrow{(f,0)} R_0$ is a projective object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. The cokernel of $R_1 \oplus R_2 \xrightarrow{(0,\xi)} R$ is $R / \text{Rej}_{\mathcal{V}}^n R$ which belongs to $\overline{\text{Gen}}V$. By Proposition 5.9 we conclude that $R_1 \oplus R_2 \xrightarrow{(f,0)} R_0$ is also a generator of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. \square

6. THE MAIN RESULT

We can now give our results, collecting what we have proved in the previous sections.

Theorem 6.1. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in $\text{Mod-}R$. The heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is equivalent to a module category if and only*

- (1) $\mathcal{X} = \text{Gen } V$ where V is a tilting R_V -module ;
- (2) V_R admits a presentation

$$0 \rightarrow \Omega \hookrightarrow R_1 \xrightarrow{f} R_0 \rightarrow V \rightarrow 0$$

with R_1 and R_0 finitely generated projective modules such that

- (a) any map $R_1 \rightarrow V$ extends to a map $R_0 \rightarrow V$,
- (b) there exists a map $R_1^{(\alpha)} \xrightarrow{g} R$ such that the cokernel of the restriction $g|_{\Omega^{(\alpha)}}$ belongs to $\overline{\text{Gen}} V$.

In such a case, $R_1 \rightarrow R_0$ is a small projective generator of the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. Denoted by S the endomorphism ring $\text{End}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})}(R_1 \rightarrow R_0)$, the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is equivalent to $\text{Mod-}S$.

Proof. It is an easy consequence of Lemma 4.1, Lemma 5.1, Proposition 5.2 and Proposition 5.9. \square

If we concentrate our attention to artinian rings the above result assume the following aspect:

Corollary 6.2. *Let R be a right artinian ring and $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in $\text{Mod-}R$. The heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is equivalent to a module category if and only if $\mathcal{X} = \text{Gen } V$ where V is a finitely presented R -module and a R_V -tilting module.*

Proof. It follows by Theorem 6.1, Proposition 5.5 and Proposition 5.11. \square

For a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\text{Mod-}R$, the tilting and cotilting notions are strictly related, as the next corollary shows.

Corollary 6.3. *Let $(\mathcal{X}, \mathcal{Y})$ be a faithful torsion pair in $\text{Mod-}R$ such that \mathcal{X} is generated by a tilting module. Then \mathcal{Y} is cogenerated by a cotilting module.*

Proof. If $\mathcal{X} = \text{Gen } V$ for a tilting module V , since the projective dimension of V is at most one and $\overline{\text{Gen}} V = \text{Mod-}R$, the assumptions of Theorem 6.1 are satisfied. In particular $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category and so, by [12], \mathcal{Y} is cogenerated by a cotilting module. \square

Remark 6.4. In [12] it is shown that the heart of a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ is equivalent to a category of modules $\text{Mod-}S$ if and only if there exists a tilting complex E^\bullet in $\mathcal{D}^b(R)$ such that the heart of the t -structure \mathcal{H}_{E^\bullet} generated by E^\bullet coincides with $\mathcal{H}(\mathcal{X}, \mathcal{Y})$. In such a case E^\bullet is a small projective generator of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ and $\text{End}(E^\bullet) \cong S$. Where to look for it, how to construct it, in terms of the torsion theory we started from, is not provided. Notice that such an E^\bullet is quasi-isomorphic to a complex $R_1 \rightarrow R_0$ satisfying conditions of Theorem 6.1.

Conversely, following [21] (more precisely, applying Theorems 2.10 and 3.8, and Corollary 3.6), we get that a complex $R_1 \rightarrow R_0$ satisfying the conditions of Theorem 6.1 is a tilting complex and the heart of the t -structure $\mathcal{H}_{R_1 \rightarrow R_0}$ generated by $R_1 \rightarrow R_0$ coincides with $\mathcal{H}(\mathcal{X}, \mathcal{Y})$.

7. APPLICATIONS

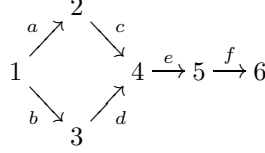
The following two examples show how strong is the property that \mathcal{X} has to be generated by a finitely presented module for $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ to be equivalent to a module category.

- Example 7.1.** (1) *In the category of abelian groups one can consider the torsion pair of torsion and torsion-free abelian groups, and that of divisible and reduced abelian groups. The heart of both these torsion pairs are not equivalent to a module category. Indeed in both the cases the torsion class is not generated by a finitely generated abelian group.*
- (2) *Let Λ be a Kronecker algebra. The closure by direct limits of the class of preinjective modules is a torsion class \mathcal{X} ; denoted by \mathcal{Y} the corresponding torsion free class, the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is not equivalent to a module category. Indeed, if \mathcal{X} is generated by a finitely presented module V , then any preinjective module would be a quotient of a direct sum of finite number of copies of V . There is no finitely presented module V with this property.*

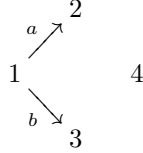
In the following example we prove as for path algebras with relations, both in the artinian and not artinian case, our algorithm permits not only to decide if the heart of a faithful torsion pair is equivalent to a module category over a ring, but in the affirmative case also to construct explicitly the ring itself.

Example 7.2. *Denote by k an algebraically closed field.*

- (1) *Let Λ be the path k -algebra given by the following quiver*



with relations $ca = db = fec = fed = 0$. Let us consider the Λ -module $V = \frac{1}{2} \oplus \frac{1}{3} \oplus 1 \oplus 4$. The subcategory $\text{Gen } V$ is a torsion class; let us denote by \mathcal{Y} the corresponding torsion free class. The ring $\Lambda_V := \Lambda / \text{Ann}_\Lambda V$ is the path algebra associated with the quiver



It is easy to verify that the finitely presented Λ -module V is a tilting Λ_V -module. Therefore, by Corollary 6.2, $\mathcal{H}(\text{Gen } V, \mathcal{Y})$ is equivalent to a module category. Since Λ is of finite representation type, $\mathcal{H}(\text{Gen } V, \mathcal{Y})$ is equivalent to a module category over an artin algebra of finite representation type Θ associated with a suitable quiver with relations. To determine it we have to find the indecomposable projective Θ -modules. Let us start constructing the small projective generator of $\mathcal{H}(\text{Gen } V, \mathcal{Y})$. Consider the following resolution of V :

$$0 \rightarrow \left(\frac{4}{5}\right)^4 \rightarrow \left(\frac{2}{4}\right)^2 \oplus \left(\frac{3}{4}\right)^2 \oplus \frac{5}{6} \rightarrow \left(\frac{1}{2} \frac{1}{3}\right)^3 \oplus \frac{4}{5} \oplus \frac{4}{6} \rightarrow \frac{1}{2} \oplus \frac{1}{3} \oplus 1 \oplus 4 \rightarrow 0$$

Since $\left(\frac{4}{5}\right)^4$ is superfluous in $\left(\frac{2}{4}\right)^2 \oplus \left(\frac{3}{4}\right)^2 \oplus \frac{5}{6}$, by Proposition 5.5, the complex with projective terms

$$\left(\frac{2}{4}\right)^2 \oplus \left(\frac{3}{4}\right)^2 \oplus \frac{5}{6} \rightarrow \left(\frac{1}{2} \frac{1}{3}\right)^3 \oplus \frac{4}{5} \oplus \frac{4}{6}$$

is a projective object in the heart. Let us consider the reject chain of Λ with respect to the family of all quotients of V :

$$\Lambda = {}_2^1 3 \oplus {}_4^2 \oplus {}_5^3 \oplus {}_6^4 \oplus {}_6^5 \oplus 6 \geq (\frac{4}{5})^2 \oplus (\frac{5}{6})^2 \oplus 6 \geq (5)^2 \oplus (\frac{5}{6})^2 \oplus 6 = \dots$$

The module $(\frac{5}{6})^4 \oplus 6$ is the projective cover of the stationary term of the reject chain. Therefore by Proposition 5.11 the complex

$$\left(\frac{2}{4}\frac{4}{5}\right)^2 \oplus \left(\frac{3}{4}\frac{4}{5}\right)^2 \oplus {}_6^5 \oplus (\frac{5}{6})^4 \oplus 6 \rightarrow ({}_2^1 3)^3 \oplus {}_6^4$$

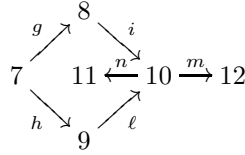
is a small projective generator of the heart. Let us decompose it as a direct sum of indecomposable projective complexes:

$$[\frac{2}{5} \rightarrow {}_2^1 3] \oplus [\frac{2}{5} \oplus {}_4^3 \rightarrow {}_2^1 3] \oplus [\frac{3}{5} \rightarrow {}_2^1 3] \oplus [\frac{4}{5} \rightarrow {}_6^5] \oplus [\frac{5}{6} \rightarrow 0]^4 \oplus [6 \rightarrow 0]$$

Forgetting the redundant repetition we get that also

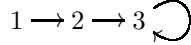
$$[\frac{2}{5} \rightarrow {}_2^1 3] \oplus [\frac{2}{5} \oplus {}_4^3 \rightarrow {}_2^1 3] \oplus [\frac{3}{5} \rightarrow {}_2^1 3] \oplus [\frac{4}{5} \rightarrow {}_6^5] \oplus [\frac{5}{6} \rightarrow 0] \oplus [6 \rightarrow 0]$$

is a small projective generator. Studying the morphisms in the heart between these projective indecomposable complexes, it is easy to verify that Θ is the path algebra associated with the quiver



with relations $mi = ml = 0$.

(2) Let Λ be the k -algebra given by the following quiver



Clearly, it is not an artinian algebra. Consider the Λ -module $V = \frac{1}{2} \oplus 1$. The subcategory $\text{Gen } V$ is a torsion class; let us denote by \mathcal{Y} the corresponding torsion free class. The ring $\Lambda_V := \Lambda / \text{Ann}_\Lambda V$ is the path algebra associated with the quiver $1 \rightarrow 2$. It is easy to verify that the finitely presented Λ -module V is a tilting Λ_V -module. Let us consider the following resolution of V :

$$0 \rightarrow \begin{matrix} \frac{3}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{2}{3} \\ \vdots \end{matrix} \rightarrow \begin{matrix} \frac{1}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{1}{3} \\ \vdots \end{matrix} \rightarrow \frac{1}{2} \oplus 1 \rightarrow 0$$

By Proposition 5.5, the complex with projective terms

$$\begin{matrix} \frac{3}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{2}{3} \\ \vdots \end{matrix} \rightarrow \begin{matrix} \frac{1}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{1}{3} \\ \vdots \end{matrix}$$

is a projective object in the heart. Let us consider the reject chain of Λ with respect to the family of all quotients of V :

$$\Lambda = \begin{matrix} \frac{1}{2} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{2}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{3}{3} \\ \vdots \end{matrix} \geq \begin{matrix} \frac{3}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{3}{3} \\ \vdots \end{matrix} \oplus \begin{matrix} \frac{3}{3} \\ \vdots \end{matrix} = \dots$$

The module $\begin{smallmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix}$ is projective. Therefore by Proposition 5.11 the complex

$$\begin{smallmatrix} 3 & 2 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 2 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \oplus \begin{pmatrix} 3 \\ 3 \\ \vdots \end{pmatrix}^3 \rightarrow \begin{smallmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ \vdots & \vdots \end{smallmatrix}$$

is a small projective generator of the heart $\mathcal{H}(\text{Gen } V, \mathcal{Y})$; thus the latter is equivalent to a module category over a ring Θ . Let us decompose our small projective generator as a direct sum of indecomposable projective complexes:

$$\begin{smallmatrix} 3 & 1 & 2 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \rightarrow \begin{smallmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 3 \\ \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 2 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \rightarrow \begin{smallmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 3 \\ \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \rightarrow 0]^3$$

Forgetting the redundant repetition we get that also

$$\begin{smallmatrix} 3 & 1 & 2 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \rightarrow \begin{smallmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 3 \\ \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 2 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \rightarrow \begin{smallmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 3 \\ \vdots & \vdots \end{smallmatrix} \oplus \begin{smallmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ \vdots & \vdots & \vdots \end{smallmatrix} \rightarrow 0]$$

is a small projective generator. Studying the morphisms in the heart between these projective indecomposable complexes, it is easy to verify that Θ is the path algebra associated with the quiver

$$4 \leftarrow 5 \rightarrow 6 \curvearrowright$$

An artin algebra Λ is *quasi tilted* if there exists a faithful splitting torsion pair $(\mathcal{X}, \mathcal{Y})$ such that any module in \mathcal{Y} has projective dimension at most one (see for instance [20]). In [19] it is showed that any quasi tilted algebra Λ of finite representation type is a tilted algebra, that is $\Lambda \cong \text{End}_\Gamma(T)$, where Γ is an hereditary algebra and T a tilting Γ -module. We get the same result applying Corollary 6.2.

Proposition 7.3. *If Λ is a quasi tilted algebra of finite representation type, then Λ is tilted.*

Proof. Let $(\mathcal{X}, \mathcal{Y})$ be a faithful splitting torsion pair in $\text{Mod-}\Lambda$ such that any module in \mathcal{Y} has projective dimension at most one. By [20] the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is an hereditary abelian category, and $\Lambda[1]$ is a tilting object in $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ with $\text{End}_{\mathcal{H}(\mathcal{X}, \mathcal{Y})} \Lambda[1] \cong \Lambda$. Therefore to get the thesis it is sufficient to prove that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is equivalent to a category of modules. By Corollary 6.2, $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is equivalent to a category of modules if and only if the torsion class \mathcal{X} is generated by a finitely presented Λ -module P which is a tilting Λ_P -module.

Let us consider the module $Q = \oplus_i^n X_i$ where $\{X_1, \dots, X_n\}$ is a complete list of non-isomorphic indecomposable modules in \mathcal{X} . Since the algebra is of finite representation type, the torsion class \mathcal{X} coincides with $\text{Add } Q$; moreover, being Q product complete, $\text{Add } Q = \text{Prod } Q$ is closed under products.

If $\text{Ext}^1(Q, Q) = 0$, we take $P := Q$. Otherwise, if $\text{Ext}^1(Q, Q) \neq 0$, let us denote by X_i and X_j two indecomposable summands of Q such that $\text{Ext}^1(X_j, X_i) \neq 0$. We claim that $\mathcal{X} = \text{Gen } Q_1 = \text{Pres } Q_1$, where $Q_1 = Q \setminus \{X_i\}$. Indeed, let $0 \rightarrow X_j \rightarrow M \rightarrow X_i \rightarrow 0$ be a non splitting exact sequence. Since in the valued quiver of Λ there are no oriented cycles (see [20]), we deduce that X_i does not belong to $\text{add } M$ and therefore M belongs to $\text{add } Q_1$. Since M generates X_i , it is $\text{Gen } Q = \text{Gen } Q_1$

and $\text{Pres } Q = \text{Pres } Q_1$. If $\text{Ext}^1(Q_1, Q_1) = 0$ we take $P := Q_1$, otherwise we repeat the same procedure. In such a way, in a finite number of steps, we will get a module $P := Q_m$ with $\mathcal{X} = \text{Gen } P = \text{Pres } P$ and $\text{Ext}^1(P, P) = 0$.

The module P is a finitely presented Λ -module; let us prove that it is a tilting Λ_P -module. By Proposition 3.2, since \mathcal{X} is closed under products, it is sufficient to prove that $\text{Gen } P = \overline{\text{Gen } P} \cap P^\perp$. If $M \in \overline{\text{Gen } P}$, there exists an exact sequence $0 \rightarrow M \rightarrow X_0 \rightarrow X_1 \rightarrow 0$ with $X_0 \in \mathcal{X}$ and $X_1 \in \text{Add } P$; if M belongs also to P^\perp this sequence splits and so M belongs to $\text{Gen } P$. Conversely, if M belongs to $\text{Gen } P = \text{Pres } P$, there exists an exact sequence $0 \rightarrow M_0 \rightarrow P_0 \rightarrow M \rightarrow 0$, where $P_0 \in \text{Add } P = \text{Prod } P$ and $M_0 \in \text{Gen } P$. Thus from the sequence $\text{Ext}^1(P, P_0) \rightarrow \text{Ext}^1(P, M) \rightarrow \text{Ext}^2(P, M_0)$, since any module in \mathcal{X} has injective dimension at most one ([20]), we conclude that $M \in P^\perp$. \square

REFERENCES

- [1] Al-Nofayee, S. *Simple objects in the heart of a t-structure*, J. Pure Appl. Algebra **213**(1) (2009), 54–59.
- [2] Angeleri Huegel, L., Šaroch, J and Trlifaj, J. *On the telescope conjecture for module categories*, J. Pure Appl. Algebra **212** (2008), 297–310.
- [3] Angeleri Huegel, L., Tonolo, A. and Trlifaj, J. *Tilting preenvelopes and cotilting precovers*, Algebr. Represent. Theory **4** (2001), 155–170.
- [4] Bazzoni, S. *Cotilting modules are pure injective*, Proc. Amer. Math. Soc. **131** (2003), 3665–3672.
- [5] Beilinson, A. A., Bernstein, J. and Deligne, P. *Faisceaux perverse*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris (1982), 5–171.
- [6] Beligiannis, A. and Reiten, I. *Homological and homotopical aspects of torsion theories* Mem. Amer. Math. Soc. **188** (2007).
- [7] Buan, A. and Krause, H. *Cotilting modules over tame hereditary algebras*, Pacific J. Math. **211** (2003), 41–60.
- [8] Colby, R. and Fuller K. *Tilting and torsion theory counter equivalences*, J. Algebra **23** (1995), 4833–4849.
- [9] Colpi, R. *Tilting in Grothendieck categories*, Forum Math. **11** (1999), 735–759.
- [10] Colpi, R., D’Este, G. and Tonolo, A. *Quasi-tilting modules and counter equivalences*, J. Algebra **191** (1997), 461–494.
- [11] Colpi, R., Fuller K. *Tilting objects in abelian categories and quasitilted rings*, Trans. Amer. Math. Soc. **359** (2007), 741–765.
- [12] Colpi R., Gregorio E. and Mantese F. *On the heart of a faithful torsion theory*, J. Algebra **307** (2007), 841–863.
- [13] Colpi R. and Gregorio E. *The heart of a cotilting torsion pair is a Grothendieck category*, preprint (2009).
- [14] Colpi, R., Trlifaj, J. *Tilting modules and tilting torsion theories*, J. Algebra **178** (2) (1995), 614–634.
- [15] Dickson, S. E. *A torsion theory for Abelian categories*, Trans. Amer. Math. Soc. **121** (1966), 223–235.
- [16] Garcia Rozas, J.R. *Covers and Envelopes in the Category of Complexes of Modules*, Chapman & Hall/CRC, London (1999).
- [17] Göbel R. and Trlifaj J. *Approximations and Endomorphism Algebras of Modules*, De Gruyter Expositions in Math. **41**, Berlin - New York 2006.
- [18] Happel, D. *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series 119, Cambridge University Press, Cambridge, 1988.
- [19] Happel D., Reiten I.. *An introduction to quasitilted algebras*, An. St. Univ. Ovidius Constantza **4** (1996), 137–149.
- [20] Happel D., Reiten I., Smalø S.O. *Tilting in Abelian Categories and Quasitilted Algebras*, Memoirs of the A.M.S., vol. **575**, 1996.

- [21] Hoshino, M., Kato, Y. and Miyachi, J. *On t -structures and torsion theories induced by compact objects*, J. Pure Appl. Algebra **167**(1) (2002), 15–35.
- [22] Jensen, C.U. and Lenzing, H. *Model Theoretic Algebra*, Algebra, Logic and Applications **2**, Gordon & Breach, Amsterdam 1989.
- [23] Mitchell, B. *The full imbedding theorem*, Amer. J. Math. **86**, (1964), 619–637.
- [24] Miyashita, Y. *Tilting modules of finite projective dimension*, Math. Z. **193** (1), (1986), 113–146.
- [25] Noohi, B. *Explicit HRS-tilting*, J. Noncommut. Geom. **3**, (2009), 223–259.
- [26] Rickard, J. *Morita theory for derived categories*, J. London Math. Soc. **39** (3), (1989), 436–456.
- [27] Reiten, I. and Ringel, C.M. *Infinite dimensional representations of canonical algebras*, Canad. J. Math. **58** (2006), 180–224.
- [28] Stenström, B. *Rings of Quotients*, Grundlehren der math. Wiss. **217**, Springer, New York (1975).
- [29] Št'ovíček, J. *All n -cotilting modules are pure injective*, Proc. Amer. Math. Soc. **134** (2006), 1891–1897.

(R. Colpi) DIP. MATEMATICA PURA ED APPLICATA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, I-35121 PADOVA ITALY
E-mail address: colpi@math.unipd.it

(F. Mantese) DIPARTIMENTO DI INFORMATICA, UNIVERSITÀ DEGLI STUDI DI VERONA, STRADA LE GRAZIE 15, I-37134 VERONA - ITALY
E-mail address: francesca.mantese@univr.it

(A. Tonolo) DIP. MATEMATICA PURA ED APPLICATA, UNIVERSITÀ DEGLI STUDI DI PADOVA, VIA TRIESTE 63, I-35121 PADOVA ITALY
E-mail address: tonolo@math.unipd.it